ON CONDITIONS OF QUASI-BRITTLE FRACTURE

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1. Problems of the theory of fracture are fundamental in the mechanics of deformable solids. Many limiting conditions formulated by various researchers for a number of models of a deformable solid, may be considered as fracture conditions.

For plastic bodies such conditions are time-independent [1]; for materials with time dependent properties, the fracture conditions may be formulated following [2 and 3].

The fracture of a number of materials may be described within the framework of an elastic body model. In this case the limiting conditions may be formulated differently.

In paper [4], carried out according to an idea of A.Iu. Ishlinskii, the fracture of an elastic body is connected with the stress reaching some ultimate value. The limiting condition $f(\sigma_{ij}) = 0$ was considered as the fracture condition for a brittle body in [5]. In [6], self-sustaining fracture is connected with the limiting value of the potential energy.

The theory of crack propagation in solids, owing its origin to Griffith, may also be considered as a theory of the fracture of elastic bodies. As a model, Griffith [7] considered an elastic body with slits S (surfaces of discontinuity of displacement). For a virtual increment in the surface of the slit δS the external forces p_i applied to the body, do the work δA, equal to the change δW ($\delta W = \delta A$) in potential energy of the body. Griffith assumed that a change in the surface of the slit δS leads to an increment in some function of the potential energy $\delta \Pi$, and he found the equilibrium condition for the slit (crack)

$$\delta \Pi = \delta W \qquad (1.1)$$

The stable state of the slit will evidently hold for $\delta \Pi > \delta W$ and the unstable state for $\delta \Pi < \delta W$. The neutral (equilibrium) state is defined by Expression (1.1). Condition (1.1) may be rewritten as

$$F - \frac{\delta W}{\delta S} = 0, \qquad F = \frac{\delta \Pi}{\delta S} \qquad (1.2)$$

Griffith interpreted the quantity $\delta \Pi$ as a change in the surface energy of the body, and defined the quantity $F = T_0$ – const as the surface tension.

Later, Irwin [8] and Orowan [9] offered another interpretation of the quantity F and connected it with the effective density of the surface energy, with the work expended in plastic deformation near the tip of the crack.

Let us also note that a discussion of various possibilities of generalizing the Griffith theory is contained in [10].

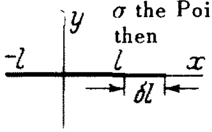
The function F for different materials may be given different interpretations (for example, the change $\delta \Pi$ may be represented as due simultaneously to a change in the surface energy and the work expended in plastic deformation during crack formation, etc.). But it should be kept in mind that the Griffith theory is one of the phenomenological theories, is a branch of continuum mechanics, and is not connected directly with the clarification of the physical mechanism of crack formation. Underlying the Griffith theory is the formulation of the relationships (1.1) and (1.2). In the general case the quantity $\delta W / \delta S$ is determined from the solution of a problem in elasticity theory. As regards the determination of the function F, it should be determined from some system of macro-experiments.

As an analogous example, we encounter the Mohr fracture theory, which is utilized to determine the strength of elastic bodies from materials with different physical mechanisms for the strength properties. However, when the Mohr theory is used, the condition of the ultimate state determined from a system of macro-tests is of interest.

The Irwin formula [11] is quite important in the theory of cracks. Irwin considered a slit of length $2l$ (Fig. 1) in the xy plane, and used asymptotic representations of the solution [12] near the tip of the slit

$$\sigma_y = \frac{N}{\sqrt{x-l}} + O(\sqrt{x-l}), \qquad v = \frac{4(1-\sigma^2)N}{E}\sqrt{l-x} + O[(l-x)^{3/2}] \tag{1.3}$$

where σ_y is the normal stress, v the displacement along the y-axis, E the elastic modulus, σ the Poisson coefficient. Moreover, Irwin found that if δl is directed along l, then



$$\delta W / \delta l = CN^2, \qquad C = 2\pi(1-\sigma^2)/E \tag{1.4}$$

It then follows from (1.2) and (1.4) that

$$N = (F_0 / C)^{1/2} \tag{1.5}$$

Fig. 1

where F_0 is the value of F referred to unit area.

According to (1.3) and (1.5), we may write that in the ultimate state

$$\sigma_y{}^\circ = \frac{(F_0/C)^{1/2}}{\sqrt{x-l}} + O(\sqrt{x-l}), \qquad v^\circ = \frac{4(1-\sigma^2)(F_0/C)^{1/2}}{E}\sqrt{l-x} + O[(l-x)^{3/2}] \tag{1.6}$$

The radius of curvature ρ of the tip of an open crack at $x = l$, evidently is:

$$\rho^\circ = {}^1/_2 N^2 = {}^1/_2 F_0 / C \tag{1.7}$$

This means that a consequence of the Irwin formula is the fact that the energy condition (1.2) may be written as either static or kinematic for the case of plane cracks [13]

$$\sigma_y = \sigma^\circ_y \qquad (N{=}No) \tag{1.8}$$

$$\rho = \rho^\circ \qquad (N{=}No) \tag{1.9}$$

In particular, it follows from the Irwin formulas (1.4) to (1.6) that if $F_0 = 0$, the stress σ_y at the edge of the crack is finite, and there is a smooth merger of the crack edges.

Khristianovich [14] proposed these properties earlier as a hypothesis. It also follows from (1.4) that the elastic energy has a stationary value in this case (this circumstance has been discussed in [15]). An extensive bibliography is devoted to questions in the mathematical theory of cracks. Let us note paper [16], devoted to the evaluation of the energy "being liberated" during the propagation of cracks, as well as [17 to 20]. Variational principles of fracture have been developed in [21], etc.

It should be noted that various singularities of the solutions are often encountered in the analysis of various problems of the mechanics of a deformed solid. For example, the unboundedness of the stress near the tips of a rigid rectangular stamp impressed in an elastic half-space, in the neighborhood of various grooves and openings (especially re-entrant angles), near concentrated forces, etc. In the theory of ideal plasticity it is possible to mention the center of the fan of characteristics at which the value of the average pressure depends on the direction of the emerging characteristics. Similar examples may be mentioned from other branches of continuum mechanics.

It is well known that such singularities of the solutions do not correspond to the behavior of real materials, and are a consequence of the assumptions defining the given model. At the same time, an analysis of these singularities is a necessary part of the mathematical investigation of the problem. In the Griffith theory of cracks (for $F \neq 0$) there are such singularities near the crack edges, in whose neighborhood, generally, some stress components become infinite.

However, this fact cannot have any essential significance in appraising the Griffith theory. The agreement between theoretical results and the data of a macro-experiment is the fundamental criterion of the value of a theory in continuum mechanics.

2. Utilizing the idea of Griffith, we consider below the condition for propagation of finite cavities in elastic bodies. We assume that there is some finite cavity in an unstressed elastic body. Furthermore, we let a system of external forced p_i be applied to the body, whereby it is deformed. For simplicity, we consider the contour of the cavity free of loadings.

Let us give the cavity a certain virtual change in volume δV (the surface δS, the length

δl, in particular cases). Let us note that in the linear theory of elasticity the boundary conditions are formulated on the undeformed surface for small deformations, and a virtual change in volume is here understood to be a change in the volume of the cavity in the undeformed state.

Such a virtual change in volume is connected with the elimination (addition) of definite constraints, hence, the external forces p_i here perform the work δA, equal to the eliminated (added) elastic energy δW.

Evidently some work of the constraints $\delta \Pi$ impedes the liberation of the elastic energy δW from the body. Hence, the condition of cavity propagation may be written as

$$\delta (\Pi - W) = 0 \tag{2.1}$$

Evidently, the stable state of the cavity holds for $\delta \Pi > \delta W$ and the unstable state for $\delta \Pi < \delta W$. The neutral (equilibrium) state is defined by (2.1).

Condition (2.1) may be rewritten as

$$F - \frac{\delta W}{\delta V} = 0, \qquad F = \frac{\delta \Pi}{\delta V} \tag{2.2}$$

The $\delta W / \delta V$ is determined on the basis of solving the elastic problem. The function F is some experimentally determined function characterizing the strength (fracture) properties of the given solid. The function F may depend on the coordinates of points of the body x_i on the direction characterized by the direction cosines α_i, the temperature, and other parameters including those dependent on the time. The expounded viewpoint generalizes the conceptions developed in the theory of propagation of slits of zero thickness (cracks).

Let us assume that the body with a cavity is in equilibrium under the effect of forces p_i. We denote the plastic energy of the original body by W_1. Let us change the cavity volume by δV. Under the same forces, the body with the altered cavity will have the elastic energy W_2. The energy ΔW is exerted on the fraction of removed volume. The desired change in elastic energy of the body will be

$$\delta W = \delta W^* + \Delta W, \tag{2.3}$$

In the case of a slit $\Delta W = 0$.

Shown in Fig. 2 is the contour L_1 of the initial state, and L_1^* of the deformed state; L_2 is the initial contour with the removed portion of the volume δV, and L_2^* is the deformed contour. The contour of the volume δV is denoted in Fig, 2a by l_1 and the deformed contour of the volume δV if it were not removed, by l_2.

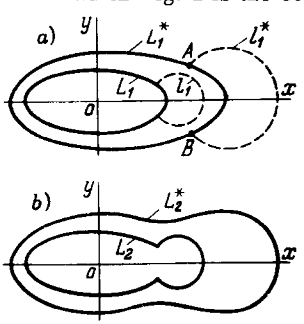


Fig. 2

According to Irwin, in order to determine δW^* it is necessary to find the work of the forces which are needed to join the contour L_2^* to the contour $L_1^* + l_1^*$ without the portion AB. According to the assumptions made, evidently that portion of the cavity (or groove) will change, for which condition (2.2) holds; hence, fracture may occur, for materials with inhomogeneous properties, not at places with the greatest state of stress. The condition of the stable state of the cavity may be written as

$$\frac{\delta W}{\delta V} < F$$

If F is a constant, then according to (2.4), the following property of the local maximum of the function W may be formulated: the development of a cavity occurs when the change in the elastic energy of a body reaches some maximum value when its volume varies.

In the general case, the development of a cavity is connected, according to (2.1), with the function $U = W - \Pi$ reaching an extremum.

Let us examine a small change in a cavity in the plane problem. Let $r = r(\varphi)$ denote the Eq. of the original contour L_1. Let us assume that the Eq. of the contour L_2 is $r_2 = r + \delta r$. Then $\delta S = \delta r \, dl \sin \chi$, where χ is the angle between δr, dl. Evidently

$$\Delta W = \int_{\delta S} \sigma_{ij} e_{ij} \, dS = \int_{L_1} (\sigma_{ij} e_{ij} \sin \chi \delta r) \, dl \tag{2.4}$$

Let $\delta u^{(n)}$ denote the difference in displacements of the contours $L_1 + l_1$, L_2, and also let $\sigma^{(n)}$ denote the stress components on the areas along l_1. According to Irwin

$$\delta W^* = \frac{1}{2}\int_{l_1} \sigma^{(n)}\,\delta u^{(n)}\,dl \qquad (2.5)$$

If $\delta W^* \ll \Delta W$, condition (2.1) is written as

$$\Delta W = \delta\Pi \qquad (2.6)$$

Let us assume that

$$\delta\Pi = K_1\delta S = K_1\int_{L_1} \sin\chi\delta r\,dl$$

$$K_1 = \text{const} \qquad (2.7)$$

We then obtain from (2.4), (2.7) and (2.6)

$$\int_{L_1} (\sigma_{ij}e_{ij} - K_1)\sin\chi\delta r\,dl = 0 \qquad (2.8)$$

The equality (2.8) evidently holds if $W^\circ = \sigma_{ij}\,e_{ij} = K_1$, or if $W^\circ < K_1$ for $\delta r = 0$.

The specific potential energy may be expressed in terms of the stress $W^\circ = \sigma_{ij}\,e_{ij} = f(\sigma_{ij})$. In this case, therefore, the energy theory of fracture reduces to the force theory considered in [4 and 5], etc. In fact, normal and tangential stress resultants are absent on the contour, hence $f(\sigma_{ij}) \sim \sigma_\varphi^2$, where σ_φ is the normal stress on areas perpendicular to the contour.

Moreover, let us assume that

$$\delta\Pi = K_2\delta L, \qquad K_2 = \text{const} \qquad (2.9)$$

where δL is the change in length of contour of the cavity. It is easy to obtain

$$\delta L = \int_{L_1} \frac{r\delta r + r'\delta r'}{\sqrt{r^2 + r'^2}}\,d\varphi \qquad (2.10)$$

where the prime denotes differentiation with respect to φ. From (2.4), (2.10), and (2.9), we obtain

$$\int_{L_1} \sigma_{ij}e_{ij}\sin\chi\,\sqrt{r^2 + r'^2}\,\delta r\,d\varphi - K_2\int_{L_1} \frac{r\delta r + r'\delta r'}{\sqrt{r^2 + r'^2}}\,d\varphi = 0 \qquad (2.11)$$

The variational Eq.

$$\int_{L_1} \left[\sigma_{ij}e_{ij}\sin\chi\,\sqrt{r^2 + r'^2} - K_2\frac{r}{\sqrt{r^2 + r'^2}} + \frac{d}{d\varphi}\left(K_2\frac{r}{\sqrt{r^2 + r'^2}}\right)\right]\delta r\,d\varphi = 0 \qquad (2.12)$$

follows from (2.11).

Fracture occurs at those points of the contour for which the integrand in the square brackets (2.12) equals zero. The case $\delta\Pi = K_3 L\delta V$, etc., may be considered analogously.

3. Let us consider some examples. Let us assume that a curvilinear slit l (Fig. 3a), terminating in some cavity at the left end, is given in the xy plane. The question is asked, in what direction does development of the right end of the slit occur (crack propagation)?

According to the concepts presented it is necessary to determine the quantity $\delta W/\delta l$ in all possible directions at the right end of the slit. We assume that either from the direct solution of the problem of elasticity theory, or by utilization of the Irwin method, etc., the magnitude of the derivative $\delta W/\delta l = \Phi(x_0, y_0, \alpha, \lambda)$ has successfully been determined for given external forces $p_i(\lambda)$, where x_0, y_0 are the coordinates of the right end of the slit; α is the angle formed by the direction δl with the t axis directed along the tangent to the slit at the right end; λ is a parameter of the change in loading.

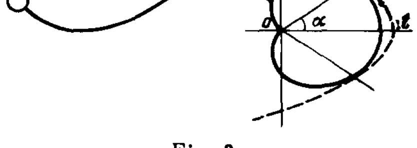


Fig. 3

The solid line in Fig. 3b pictures the graph of the quantity $\delta W/\delta l$ in tn-axes. The dashes show the curve of $F(x_0, y_0, \alpha)$. The curve of F is fixed for any point x_0, y_0. Let us note that if $F \equiv$ const, then the corresponding curve in the tn plane will be a circle with center at the origin.

The curve Φ is tangent to the curve F for some λ. Then directions of crack development correspond to rays drawn from the origin to the point of tangency of the Φ, F curves. If tangency occurs at several points simultaneously as is shown in Fig. 3b, then crack development immediately occurs in several directions, i.e., a point of crack bifurcation holds. If tangency occurs at a point on the t-axis ($n = 0$), the crack is propagated in a direction tangent to its end.

It is evident that if both ends of the crack may develop, the problem of determining Φ should be solved independently for both ends, and that value of λ must be determined for which tangency of the curves Φ and F will first be achieved at one of the ends. Furthermore considering the crack development an analogous problem for the slit $l + \delta l$, etc., should be solved. The problem of the development of more complex slits is similarly posed.

As another illustration, let us mention the origin of cracks under a rigid rectangular stamp impressed in an elastic half-space (Fig. 4). Let us assume that the cracks form near the edges of the stamp. Giving the virtual slits δl, the quantity $\Phi(\pm a, 0, a, p_0)$ should be determined where a is half the length of the stamp, and p_0 is the average pressure of the stamp. Let the curve Φ be determined and its graph pictured in Fig. 4b. If the curve Φ touches the curve $F(\pm a, 0, a)$ at a point of the t-axis ($n = 0$), the crack originates in a direction parallel to the y-axis, etc.

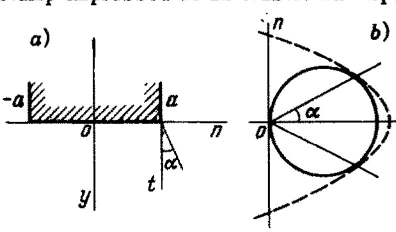


Fig. 4

Finally, let us consider the development of a circular cavity of radius a in an elastic space (plane deformation) compressed at infinity by a uniform pressure p_0 (Fig. 5). For simplicity, we assume the material to be incompressible. In polar coordinates [22]

$$\sigma_r, \sigma_\theta = -p_0\left(1 \mp \frac{a^2}{r^2}\right), \qquad u = -\frac{3p_0a^2}{Er} \tag{3.1}$$

where σ_r, σ_θ are respectively, the radial and tangential stress components, u the radial displacement, and r the moving radius. It is easy to determine

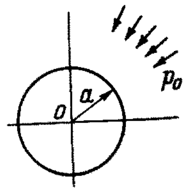


Fig. 5

$$W = 2\pi \int_a^\infty (\sigma_r - \sigma_\theta)\, e_r r\, dr = \frac{6\pi p_0^2 a^2}{E} \tag{3.2}$$

from which follows

$$\delta W = \frac{12\pi p_0^2 a\delta a}{E}, \qquad \frac{\delta W}{\delta V} = \frac{6p_0^2}{E}, \qquad \delta V = 2\pi a\delta a \tag{3.3}$$

For example, if $F = Ka$, where K is a constant, then equilibrium development of the cavity will occur, according to (2.1), for

$$p_0 = \sqrt{1/_6 EKa} \tag{3.4}$$

Other cases may be examined analogously. Let us show that in this case $\delta W^* \ll \Delta W$. From (2.4), (3.1) it follows

$$\Delta W = \int_0^{2\pi} (\sigma_\rho - \sigma_\theta)\, e_\rho a\delta a\, d\varphi = \frac{12\pi p_0^2 a\delta a}{E} \tag{3.5}$$

From (3.5), (3.3) it follows that $\delta W = \Delta W$. It is easy to obtain

$$\delta W^* = \frac{6p_0^2(\delta a)^2}{E} \tag{3.6}$$

Hence, within the scope of the elucidated considerations, it turns out to be possible to combine the approaches to problems of the theory of crack formation and development and the development of finite cavities in an elastic body.

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